Riemann-Roch inequality for ample divisors

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Riemann-Roch type inequality for ample divisors

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Let V^n be a complete non-singular variety and X a divisor on V. We let $\ell(X) = h^0(V, \underline{O}(X))$ for the sake of simplicity. Assuming ~~that~~ X is ample on V, we are interested in finding an inequality of the form

$$\ell(rX) \geq \chi(V,\underline{O}(rX)) - Q(r),$$

where r is a positive integer and $Q(x)$ a polynomial with the following properties:

(a) $Q(x)$ depends only on n and $\deg Q(x) < n$;

(b) Each coefficient of $Q(x)$ is a polynomial of the coefficients of $\chi(V, \underline{O}(mX))$ and $I(K_V^{(2)}, X^{(n-2)}) = \eta$ over the field of rational numbers (K_V denotes a canonical divisor on V).

To simplify notations, we shall let $d = X^{(n)}$ and $\xi = I(K_V, X^{(n-1)})$.

Essentially, terminology and conventions are based on Weil's "Foundations of Algebraic Geometry". Basic results in it and in Grothendieck's "Eléments de géométrie algébriques" will be used freely. Aside from these, Kodaira vanishing theorem is essential in § 1 (cf.Kodaira,"On a differential-geometric method in the theory of analytic stacks",Proc.,Nat.Acad.Sci.,U.S.A,1953),together with resolution of singularities (cf.Hironaka,"Resolution of singularities of an algebraic variety over a field of characteristic 0",I-II,Annals of Math.,1964). Results in § 2 are based in part on the following articles: Matsusaka-Mumford,"Two fundamental theorems on deformations of polarized varieties",Amer. J.,1964;Matsusaka,"Polarized varieties with a given Hilbert polynomial",Amer.J.,1972. The contents of § 5 (Appendix),together with an application can be found in the following articles:

Matsusaka,"On polarized varieties of dimension 3",I-II-III, Amer.J.,1979,1980,forthcoming.

§ 1 Choice of u

To begin with, we start with simple exact sequences of modules of rational functions on V we are going to deal with. Let us assume for a moment that u is a positive integer with

$$\chi(K_V + uX) > 1.$$

We consider the complete linear system $\Lambda(K_V + uX)$ defined by $K_V + uX$. In general, when Λ is a linear system, we shall denote by Λ_{red} the linear system which is obtained from Λ by removing the fixed part of it, and call it a reduced linear system.

Let us assume further that a suitable linear subsystem Λ of $\Lambda(K_V + uX)$ satisfies the following conditions:

(i) $\chi(rX + K_V + uX) = \chi(V, \underline{O}(rX + K_V + uX))$ for all positive integers r;

(ii) $\chi(K_V + uX) > 1$ and a general divisor W of Λ_{red} is absolutely irreducible;

(iii) A rational map f of V into a projective space defined by Λ does not decrease the dimension of V and does not collaspe (generically) each subvariety D of codimension 1;

(iv) The linear system $Tr_D \Lambda_{red}$ has reasonably big dimension for such D;

(v) When U^2 is the intersection-product of $n - 2$ general divisors of the complete linear system $\Lambda(mX)$ for large m (it is absolutely irreducible and non-singular), $I(K_V+uX+D,D,U) \geqslant 0$ for each such D and $K_V + uX$ induces on U a numerically effective divisor (i.e.arithmetically effective divisor).

We denote by $\sum_1^w D_i$ the fixed part of Λ and use the convention that $W = D_o$. For each s, we can find a V-divisor T_s which is linearly equivalent to $rX + K_V + uX - \sum_o^s D_i$, such that every component of $T_{s-1} \cap D_s$ is proper on V and simple on D_s. Note that $T_{s-1} - D_s \sim T_s$. Then we have the following exact sequences of modules:

$$0 \longrightarrow L(T_{s-1} - D_s) \longrightarrow L(T_{s-1}) \longrightarrow Tr_{D_s} L(T_{s-1}) \longrightarrow 0,$$

where $L(A)$ denotes the module of rational functions g with $div(g) + A \succ 0$. When we denote also by $\chi(D_s; T_{s-1}.D_s)$ the dimension of $H^o(D_s; \underline{O}(T_{s-1}.D_s))$, we get from the above the following inequalities:

$$\chi(T_{s-1} - D_s) - \chi(T_{s-1}) + \chi(D_s; T_{s-1}.D_s) \geqslant 0.$$

When we add these up, we find that

(1) $$\chi(rX) - \chi(rX + K_V + uX) + \sum \chi(D_s; T_{s-1}.D_s) \geqslant 0.$$

This is the basic inequality we are going to deal with. First we shall deal with the problem of finding a suitable u so that the five conditions above are satisfied.

Conditions (iii),(iv),(v) are essential for $n > 2$, since we estimate $\chi(D_s; T_{s-1}.D_s)$ in terms of intersection-numbers involving a general divisor of $Tr_{D_s} \Lambda_{red}$. Moreover, we want to select u as small as possible. Therefore, <u>we shall assume from now on that the characteristic of the universal domain is</u> 0.

By our assumption above for the characteristic, the condition (i) is trivially satisfied by Kodaira vanishing theorem for all positive integer u. As to the first half of (ii), we note that the following fact is true.

Lemma 1.1

Let V^n be a projective non-singular variety, K_V a canonical divisor on V and Y a V-divisor such that $Y^{(n)} > 0$ and that the complete linear system $\Lambda(mY)$ determined by mY has no base point for large m. Then when u_o is an arbitrary non-negative integer, there is at least one u such that $1 \leqslant u \leqslant n+1$ and

$$\chi(V, \underline{0}(K_V + u_oY + uY)) \geqslant 1.$$

This is an immediate corollary of a generalization of Kodaira vanishing theorem.

Corollary 1

When we let $T = K_V + u_oY + (n+1)Y$, $\chi(mT) \geqslant 1$ for all large values of m.

Corollary 2

Let D be a subvariety of V of codimension 1. If $I(D, Y^{(n-1)}) > 0$ and D itself is non-singular,

$$I(K_V + nY + D, D, Y^{(n-2)}) \geqslant 0.$$

This follows at once from the above and from the adjunction formula.

Corollary 3

In the above Corollary 2, D may have singularities.

Intersect $n - 2$ general members of the complete linear system $\Lambda(mY)$ for large m, which is non-singular by Bertini's theorem. Then desingularize D as an imbedded algebraic variety in V^n, using succession of monoidal transformations with non-singular centers, pull back the above intersection (which is non-singular), apply the result of Corollary 2 and use basic results on exceptional curves of the first kind. (The original proof of the author was somewhat longer, The above simplification was suggested by Reid).

As an application, we get the following lemma.

Lemma 1.2

Let V^n be a non-singular projective variety and X an ample V-divisor, both rational over an algebraically closed field k. Let $X_m^{(1)}, \ldots, X_m^{(n-2)}$ be an independent generic divisors of the complete linear system $\Lambda(mX)$ for large m and let U^2 be the intersection-product of these $n - 2$ divisors. Then $K_V + uX$, for $u \geqslant n + 1$, induces on U an arithmetically effective (i.e. numerically effective) divisor on U.

Let $Y = K_V + uX$ and $Y' = Y.U$. As $\chi(tY) > 0$ for large t, $\chi(tY') > 0$ for large t and for much larger m. When that is so, $\chi(tY') > 0$ for large t when a large fixed m is fixed.

When C is a curve on U which is not a fixed component of $\Lambda(tY')$, $I(Y',C) \geqslant 0$. Also when $C^{(2)} \geqslant 0$, we see that $I(Y',C) \geqslant 0$ by standard arguments.

Assume now that C is a fixed component of $\Lambda(tY')$ and that $C^{(2)} < 0$. Then C is a fixed component of $Tr_U \Lambda(tY)$. From the definition of U and from the fact that $\Lambda(tY)$ has a structure of an algebraic variety over k, it follows that $\Lambda(tY)$ has a fixed component F and C is a component of $F \cap U$. When that is so, $C = F.U$ from the definition of U and from the fact that F is defined over k. By Corollary 3 above, $I(K_V + uX, F, X^{(n-2)}) \geqslant -I(F^{(2)}, X^{(n-2)})$ and the latter is $-(1/m^{n-2}).C^{(2)}$. Hence the left hand side of the above is non-negative. This proves our lemma.

§ 2 The choice of u(continued)

Here we start reviewing some notions. Let U^n be a variety defined over a field k and M a finitely generated module of rational functions on V which has a basis over k. Let Λ be the reduced linear system of U-divisors defined by the module M. Set theoretically, Λ is the set of divisors $\operatorname{div}(f) + Z$, $f \in M$, where $Z = \text{L.C.M}(\operatorname{div}(f)_\infty, f \in M)$. Moreover, Λ has a structure of an algebraic variety over k. Let f be a non-degenerate rational map of U into a projective space P defined by M, and we take f so that it is defined over k. Let $H_1, \ldots, H_s$ be s independent generic hyperplanes in P over k (i.e. we assume that the set of coefficients of defining equations of the H_i are algebraically independent over k). Then the intersection-theoretically defined cycle $f^{-1}(H_1 \ldots H_s)$ on U will be called a <u>variable</u> (n-s)-<u>cycle of</u> Λ over k. It is uniquely determined up to k-isomorphisms.

Let Y be a k-rational U-divisor such that rY is locally principal for some integer r. $I(Y^{(n-s)}, f^{-1}(H_1 \ldots H_s))$ is well defined if $\dim \operatorname{Im}(f) \geqslant s$. If $\dim \operatorname{Im}(f) < s$, we define the above number to be 0. Then <u>the number is independent of base extensions of</u> k. This number will be denoted by the symbol

$$I(Y^{(n-s)}, \Lambda^{[s]}).$$

Another way of defining the number is as follows. Let $Z_1, \ldots, Z_s$ be independent generic divisors of Λ over k. Let the A_i be the proper components of $Z_1 \cap \cdots \cap Z_s$ on U of the multiplicities a_i such that the k-closure of $\operatorname{supp}(A_i) = U$. Then $\sum a_i A_i$ is a variable (n-s)-cycle of Λ over k.

When U is non-singular and complete, $\Lambda^{[s]}$ can be regarded as the numerical equivalence class defined by $\sum a_i A_i$. With respect to the concept of variable cycles, we shall list some of the known results and trivial consequences of them. Proofs or essential part of them can be found in writer's articles ("Two fundamental theorems ...", "Polarized varieties with a given Hilbert Polynomials").

Lemma 2.1

Let U^n be a projective variety, non-singular in codimension 1 and Y a U-divisor such that the complete linear system $\Lambda(mY)$ contains a hypersurface section of U for large m. Let Λ be a linear system of U-divisors, defined by a finitely generated module M of rational functions on U and f a non-degenerate rational map of U into a projective space defined by M. Assume that $\dim \mathrm{Im}(f) = s$. Then

$$\dim \Lambda = \dim M - 1 \leqslant I(Y^{(n-s)}, \Lambda^{[s]}) + s - 1.$$

Lemma 2.2

With the same notations and assumptions of Lemma 2.1, let Λ' be a linear subsystem of Λ. Then

$$I(Y^{(n-s)}, \Lambda'^{[s]}) \leqslant I(Y^{(n-s)}, \Lambda^{[s]}).$$

Lemma 2.3

With the same notations and assumptions of Lemma 2.1, assume further that $\dim \mathrm{Im}(f) = n$. Let A be a subvariety of U of codimension 1 such that its proper image $f[A]$ has dimension $s < n - 1$. When f' is the restriction of f on A, a general fibre B of f' has dimension $n - 1 - s$. Moreover, we have the following results. (a) Let Λ^* be the set of all those $Z \in \Lambda$ with $\mathrm{supp}(Z) \supset \mathrm{supp}(B)$. Then Λ^* is a linear subsystem of Λ of codimension 1. (b)

$$I(Y^{(n-1-s)}, \Lambda^{*[s+1]}) \leqslant I(Y^{(n-1-s)}, \Lambda^{[s+1]}) - I(Y^{(n-1-s)}, B).$$

Lemma 2.4

Let V^n be a non-singular projective variety and X an ample V-divisor. Assume that $\chi(K_V + uX) \geqslant 1$ and $\Lambda = \Lambda(K_V + uX)$. Then

$$I(X^{(n-i)}, \Lambda^{[i]}) \leqslant d(u + \xi/d)^i.$$

Remark Note that $\chi(K_V + uX) \geqslant 1$ implies that $u + \xi/d > 0$.

From Lemmas 2.1 and 2.4, we get

Lemma 2.5

With the same notations and assumptions of Lemma 2.4, let Λ' be a linear subsystem of Λ, satisfying $\dim \Lambda' \geq d(u+\xi/d)^{n-1} + n-1$. Let g be a non-degenerate rational map of U into a projective space defined by Λ' (i.e. by a definning module of Λ'). Then

$$\dim \mathrm{Im}(g) = n,$$

provided that $u + \xi/d \geq 1$.

Going back, as in Lemma 2.5, to our original pair (V, X), we define a series of polynomials as follows.

$$\nu(x) = x + \xi/d\ ;$$

$$R_n(x) = \Sigma_1^{n-1}\, d.\nu(x)^i;$$

$$R'_n(x) = d.\ \nu(x)^n/(n! + 1) + d.\nu(x);$$

$$R''_n(x) = (d + 1/(n!+1))\nu(x)^{n-1} + n;$$

$$R^*_n(x) = R_n(x) + R'_n(x) + R''_n(x).$$

Lemma 2.6

Let $\Lambda = \Lambda(K_V + uX)$ and assume that $u + \xi/d \geq 1$ and that $\dim \Lambda - R^*_n(u) > 0$. Then there is a linear subsystem Λ' of Λ with the following properties: (a) A non-degenerate rational map f of V into a projective space defined by Λ' is such that $\dim \mathrm{Im}(f) = n$; (b) f has no contractible subvariety of V of codimension 1; (c) When D is a subvariety of V of codimension 1,

$$\dim \mathrm{Tr}_D \Lambda'_{red} \geq \nu(u)^{n-1}/(n!+1) + 1.$$

Remark We say that a subvariety A of V is contractible by f, if f is defined at some point of A and the induced rational map f' of f on A is not generically finite)

<u>Outline of a proof</u> We define two types of operations on linear subststems of Λ. Denote by v (resp. w) the number of type I (resp. type II) operations. We define $\Lambda = \Lambda(0,0)$ and denote by $\Lambda(v,w)$ the result of applying the type I operations v-times and the type II operations w-times. Assume that $\Lambda(v,w)$ has been defined as a linear subsystem of $\Lambda(v',w')$ with $v + w = v' + w' + 1$ and that $\dim \Lambda(v,w) > d.\nu(u)^{n-1} + n - 1$. This means in particular that a non-degenerate rational map $f_{v,w}$ of V into a projective space, defined by $\Lambda(v,w)$, satisfies (a).

Assume that $f_{v,w}$ has a contractible subvariety of codimension 1. We apply Lemma 2.3 and get a linear subsystem of $\Lambda(v,w)$ which will be called a type I operation and the resulting linear system will be denoted by $\Lambda(v+1,w)$. We have

$$(2.1) \qquad \dim \Lambda(v+1,w) = \dim \Lambda(v,w) - 1;$$

$$\Sigma_1^{n-1} I(X^{(n-i)}, \Lambda(v+1,w)^{[i]}) \leqslant \Sigma_1^{n-1} I(X^{(n-i)}, \Lambda(v,w)^{[i]})-1.$$

When there is no such subvariety of codimension 1, we look for a subvariety D of V of codimension 1 such

$$\dim \mathrm{Tr}_D \Lambda(v,w)_{red} \leqslant \nu(u)^{n-1}/(n! + 1).$$

When there is such D, we get

$$(2.2) \qquad \dim (\Lambda(v,w)_{red} - D) \geq \dim \Lambda(v,w) - \nu(u)^{n-1}/(n!+1) - 1.$$

From $\Lambda(v,w)_{red} - D$, we recover a linear subsystem $\Lambda(v,w+1)$ of $\Lambda(v,w)$ with at least one additional fixed component D. From (2.1), (2.2) and from the comment above, we find that

$$\dim \Lambda(v,w) \geq \dim \Lambda - v - w.(\nu(u)^{n-1}/(n!+1) + 1),$$

$$\Sigma_1^{n-1} I(X^{(n-i)}, \Lambda(v,w)^{[i]}) \leqslant \Sigma_1^{n-1} I(X^{(n-i)}, \Lambda^{[i]}) - v - w,$$

as $\Lambda(v,w)$ has been obtained from successively by these two operations. Moreover, since the left hand side of the second formula is non-negative, $v + w \leqslant R_n(u)$ by Lemma 2.4, and

$v \leqslant R_n(u)$ in particular. As w is at most the number of total fixed component of a linear subsystem of Λ, $w \leqslant I(K_V + uX, X^{(n-1)})$ $= d.\iota(u)$. When that is so,

$$\dim \Lambda(v,w) \geqslant \dim \Lambda - R_n(u) - R'_n(u) > R''_n(u).$$

This shows that $\dim \Lambda(v+1,w)$ or $\dim \Lambda(v,w+1) > d.\iota(u)^{n-1}$ $+ n - 1$. Our lemma follows from this at once.

Proposition 2.1

Let u be a positive integer satisfying

$$u \geqslant n + 2,$$

$$\chi(V, \underline{O}(K_V + uX)) - R^*_n(u) > 0.$$

There is a linear subsystem Λ' of $\Lambda(K_V + uX)$ which satisfies our requirements (i),(ii),(iii),(iv) and (v). Moreover,

$$du^2 + 2\xi u + \eta > 0$$

for such u, and Λ' satisfies (a),(b),(c) of Lemma 2.6.

Note that the last statement follows from Lemma 1.2.

§ 3

Even though $\Lambda(K_V + uX)$ may not satisfy our five requirements in § 1, we have found that a suitable linear subsystem of it satisfies all those conditions. From now on, we shall fix one such linear subsystem and call it Λ.

In particular, $W = D_o$ denotes a general divisor of Λ_{red} and $\sum_1^w D_i$ denotes the fixed part of Λ. T_s is a V-divisor, linearly equivalent to $K_V + (u+r)X - \sum_o^s D_i$ such that $T_s \cap D_{s+1}$ is a proper intersection on V, where r is an arbitrary positive integer. We are going to estimate

$h^o(D_s; \underline{O}(T_{s-1} . D_s)) = \chi(D_s; T_{s-1} . D_s)$ and their sum.

In order to do it, first we need the following lemma.

Lemma 3.1

Let U^n be a normal projective variety and A a U-divisor such that mA is linearly equivalent to a hypersurface section of U for some large m. Let A be a U-divisor such that $\Lambda(mZ)$ is free from a fixed component and is not composed of a pencil for large m. Assume that $v.A^{(n)} \geqslant I(A^{(n-1)}, Z)$ for some positive integer v. Then for large t, $\Lambda(vt(n!+1)A)$ contains $\Lambda(tZ) + F$ as a linear subsystem, where F is a suitable fixed positive divisor on U.

Proof To simplify notations, we let t be an arbitraily large positive integer, divisible by $n! + 1$ and $t'(n! + 1) = t$. We have

$$\chi(tvA) = e_o t^n + e_1 t^{n-1} + \dots + e_n , \quad e_o = v^n A^{(n)}/n!$$

By our assumption, $v^n A^{(n)} \geqslant I((vA)^{(n-1)}, Z)$ and hence

$$e_o \geqslant (1/n!) I((vA)^{(n-1)}, Z) > (1/(n!+1)) I((vA)^{n-1)}, Z).$$

Let B be a general divisor of $\Lambda(t'Z)$. B is absolutely irreducible by Bertini's theorem and

$$(*) \quad 0 \leqslant \chi(tvA) - I((tvA)^{(n-1)}, B) - n + 1.$$

We have the following exact sequence

$$0 \longrightarrow L(tvA - B) \longrightarrow L(tvA) \longrightarrow Tr_B L(tvA) \longrightarrow 0.$$

Hence $\dim Tr_B L(tvA) = \dim L(tvA) - \dim L(tvA - B)$. If $L(tvA - B)$ is empty, we get $\dim Tr_B L(tvA) = \dim L(tvA)$. On the other hand, $\dim Tr_B L(tvA) \leqslant I((tvA)^{(n-1)}, B) + n - 1$ by Lemma 2.1. This, together with the above equality leads to a contradiction with (*).

Lemma 3.2 - Definition

$$I(W, D_i, X^{(n-2)}) \leqslant du^2 + 2\xi u + \eta = \tau \quad .$$

Proof We may assume that W and the D_i intersect properly on V. Then $I(W, D_j, X^{(n-2)}) \geqslant 0$ for all j. It follows that $I(W, D_i, X^{(n-2)}) \leqslant I(W, K_V + uX, X^{(n-2)})$. By our choice of u and Cor.3, Lemma 1.1, $I(D_j, K_V + uX, X^{(n-2)}) \geqslant 0$ for all j. When that is so, $I(W, D_i, X^{(n-2)}) \leqslant I((K_V + uX)^{(2)}, X^{(n-2)})$. Our Lemma follows at once from this.

Lemma 3.3

Neither Λ_{red} nor $Tr_D \Lambda_{red}$ is composed of a pencil for an arbitrary subvariety D of codimension 1 of V for $n > 2$.

Proof As a non-degenerate rational map of V into a projective space defined by Λ maps V on a variety of dimension n (cf. Lemma 2.6), Λ cannot be composed of a pencil.

By the same Lemma, the same map maps D on a variety of dimension $n - 1$ generically. Therefore $Tr_D \Lambda_{red}$ cannot be composed of a pencil for $n > 2$.

The above Lemma implies that W is absolutely irreducible and the same is true for a general divisor of $(Tr_D \Lambda_{red})_{red}$ by a theorem of Bertini.

Lemma 3.4

There is a normal variety D^*_s and a birational morphism g_s of D^*_s on D_s with the following properties.

(a) Denote by $g_s^{-1}(Tr_{D_s}\Lambda_{red}) = \Lambda^*_s$ the linear system on D^*_s, the smallest linear system containing the $g_s^{-1}(A)$ for general A in $Tr_{D_s}\Lambda_{red}$. Then $\Lambda^*_{s,red}$ has no base point.

(b) $\chi(D_s; T_{s-1}.D_s) \leqslant \chi(D^*_s; g_s^{-1}(T_{s-1}.D_s))$, the latter being defined by $\dim \Lambda(D^*_s; g_s^{-1}(T_{s-1}.D_s)) + 1$ here.

(c) When Z^*_s denotes a general divisor of $\Lambda^*_{s,red}$, $tZ^* + F$ is linearly equivalent to $((n-1)!+1)t\tau g_s^{-1}(X.D_s)$ for large t and for a suitable $F > 0$.

Proof (ii) is obvious for all such (D^*_s, g^*_s). To make the notations simpler, we shall omit indices s.

Let f be a non-degenerate rational map of V into a projec-space by Λ_{red}. Then the induced map f' by f on D is obviously defined by $Tr_D\Lambda_{red}$. Let G be the graph of f' and (D^*, ρ) be the normalization of D. Let π be the projection of G on D. Both ρ and π are birational morphisms.

As is well known and easy to see, $\pi^{-1}(Tr_D\Lambda_{red})$ defined as in (a) is a linear system of divisors on G such that its reduced part (i.e. the linear system - the fixed part) has no base point. Let $g = \pi \circ \rho$ and we take (D^*, g) for our purpose.

Then it is clear that $g^{-1}(Tr_D\Lambda_{red}) = \Lambda^*$ is a linear system on D^* such that Λ^*_{red} has no base point.

Let Z^* be a general divisor of Λ^*_{red} and $X^* = g^{-1}(X.D)$. Then we have

$$I(Z^*, X^{*(n-2)})_{D^*} = I(g(Z^*),(X.D)^{(n-2)})_D \leqslant I(W.D, (X.D)^{(n-2)})_D$$
$$= I(W, D, X^{(n-2)}) \leqslant \tau \leqslant \tau I(D, X^{(n-1)}) \text{ (cf.Lemma 3.2)}$$
$$= \tau(X.D)^{(n-1)}_D = \tau X^{*(n-1)}.$$

Therefore, (c) follows from Lemma 3.1.

Using our Lemma 3.4 we translate the problem of estimating $\chi(D_s; T_{s-1}.D_s)$ to that of $\chi(D^*_s; g_s^{-1}(T_{s-1}.D_s))$. For this purpose we have to carry out the following two programs.

(I) Determine ν_i, as small as possible, so that (on D^*_s)

$$I(Y^*_s - \nu_i Z^*_s, Z^*_s{}^{(i)}, X^*_s{}^{(n-i-2)}) < 0,$$

where Y^*_s is a general divisor of $\Lambda(D^*_s; g_s^{-1}(T_{s-1}.D_s))_{red}$ and $X^*_s = g_s^{-1}(X.D_s)$.

(II) Find relations between $I(Y^*_s, Z^*_s{}^{(n-2)})_{D^*_s}$ and $I(K_V + (u+r)X, D_s, X^{(n-2)})$.

A consequence of (I) is as follows. For the sake of simplicity, let us omit indices s. For an arbitrary positive integer v, denote by $Z^*(v)$ a general divisor of the complete linear system $\Lambda(vZ^*)$, which has no base point. Then we get first an inequality $\chi(Y^*) \leqslant \chi(Z^*(\nu_1); Y^*.Z^*(\nu_1))$, using the exact sequence

$$0 \longrightarrow L(Y^* - Z^*(\nu_1)) \longrightarrow L(Y^*) \longrightarrow Tr_{Z^*(\nu_1)} L(Y^*) \longrightarrow 0.$$

On $Z^*(\nu_1)$, $I(Y^*.Z^*(\nu_1) - \nu_2 Z^*.Z^*(\nu_1), (X^*.Z^*(\nu_1))^{(n-3)})$

$$= I(Y^* - \nu_2 Z^*, Z^*(\nu_1), X^{*(n-3)})_{D^*} = \nu_1 . I(Y^* - \nu_2 Z^*, X^{*(n-3)})_{D^*} < 0.$$

Hence, using a similar exact sequence as above, we find that

$$\chi(Z^*(\nu_1); Y^*.Z^*(\nu_1)) \leqslant \chi(Z^*(\nu_1).Z^*(\nu_2); Y^*.Z^*(\nu_1).Z^*(\nu_2)).$$

Continuing this, and denoting by $\prod_1^{n-2} Z^*(\nu_i)$ the intersection product of the $Z^*(\nu_i)$, we see that

$$(**) \quad \chi(Y^*) \leqslant \chi(\textstyle\prod_1^{n-2} Z^*(\nu_i); Y^*.\prod_1^{n-2} Z^*(\nu_i))$$
$$\leqslant \deg Y^*.\textstyle\prod_1^{n-2} Z^*(\nu_i) + 1 = \prod_1^{n-2} \nu_i . I(Y^*, Z^{*(n-2)}) + 1.$$

Therefore, the purpose of (II) must also be clear.

§ 4 Solutions of (I) and (II).

We select indices in $W + \sum_1^w D_i$, $W = D_o$, so that the followconditions are satisfied:

$$W = D_o ,$$
$$I(\Sigma_o^{s-1} D_i ,\ D_s ,\ X^{(n-2)}) > 0 \quad \text{for all} \quad i.$$

This is possible by Lemma 1.2, Proposition 2.1 and by Kodaira's Lemma.(Kodaira,"Pluricanonical systems on algebraic surfaces", J.Math.Soc.,Japan,1968)

Lemma 4.1

Assume that $\Lambda(D^*_s;\ g_s^{-1}(T_{s-1}.D_s)) \neq \emptyset$. Let Y^*_s be a general divisor of the reduced linear system obtained from the linear system above and $X^*_s = g_s^{-1}(X.D_s)$. Then

$$I(Y^*_s,\ X^*_s{}^{(n-2)}) \leqslant I(D_s,\ K_V + (u+r)X,\ X^{(n-2)})$$
$$\leqslant I(K_V + uX,\ K_V + (u+r)X,\ X^{(n-2)}).$$

Proof For the sake of simplicity, we shall omit indices s. Since X is an ample divisor on V , we get the following series of equalities and inequalities:

$$I(Y^*,\ X^{*(n-2)})_{D^*} \leqslant I(g^{-1}(T_{s-1}.D_s),\ X^{*(n-2)})_{D^*}$$
$$= I(T_{s-1}.D_s\ ,\ (D.X)^{(n-2)})_D = I(T_{s-1},\ D_s,\ X^{(n-2)})$$
$$= I(D_s,\ K_V + (u+r)X - \Sigma_o^{s-1} D_i,\ X^{(n-2)})$$
$$\leqslant I(D,\ K_V + (u+r)X,\ X^{(n-2)}) \leqslant I(K_V + uX,\ K_V + (u+r)X,\ X^{(n-2)}).$$

The last portion follows from Lemma].2 since $u + r > n + 1$.

Lemma 4.2

With the same notations and assumptions of Lemmas 3.4 and 4.1,

$$I(Y^*_s, Z^*_s{}^{(i)}, X^{*(n-i-2)})_{D^*_s}$$
$$[((n - 1)! + 1)\tau]^i I(K_V + uX,\ K_V + (u+r)X,\ X^{(n-2)}).$$

Proof We shall omit s for the sake of simplicity. By Lemma 3.4, (c), $((n - 1)! + 1)t\tau X \sim tZ^* + F$, $F > 0$ for large t. Moreover, by the definition of Y^*, we may assume that no component of F is a component of Y^*. It follows that

$$t.I(Y^*, Z^{*(i)}, X^{*(n-i-2)}) \leqslant I(Y^*, Z^{*(i-1)}, tZ^* + F, X^{*(n-i-2)})$$
$$\leqslant ((n-1)!+1)t\tau I(Y^*, Z^{*(i-1)}, X^{*(n-i-1)}).$$

This implies $I(Y^*,Z^{*(i)},X^{*(n-i-1)}) \leqslant [((n-1)!+1)\tau]^i I(Y^*,X^{*(n-2)})$.

The rest of our lemma follows from this and from Lemma 4.1.

<u>Lemma</u> 4.3

Keeping the same notations of Lemma 3.4, Let $g_s^{-1}(Tr_{D_s} \Lambda_{red}) = \Lambda^*_s$ and Z^*_s a general divisor of $\Lambda^*_{s,red}$. Then we have the following results:

(a) $\Lambda^{*\,[n-1]}_s \geqslant (u + \xi/d)^{n-1}/(n!+1) - n + 3$;

(b) $[((n-1)!+1)\tau]^i I(Z^{*\,(n-i-1)}_s, X^{*\,(i)}_s) \geqslant Z^{*\,(n-1)}_s$
$$\geqslant (u + \xi/d)^{n-1}/(n!+1) - n + 3.$$

Proof By our choice of u and Λ (cf.Proposition 2.1),

$$\dim Tr_D \Lambda_{red} \geq \tau(u)^{n-1}/(n!+1) + 1,$$

where the indices s are omitted as usual. By Lemma 2.1, $\dim Tr_D \Lambda_{red} \leqslant (Tr_D \Lambda_{red})^{[n-1]} + n - 2$. It follows that

$$(Tr_D \Lambda_{red})^{[n-1]} \geqslant \tau(u)^{n-1}/(n!+1) - n + 3.$$

Since we have the obvious equality $(Tr_D \Lambda_{red})^{[n-1]} = \Lambda^{*[n-1]}$.

(a) is thereby proved.

By Lemma 3.4, $((n-1)!+1)t\tau X^* \sim tZ^* + F$, $F > 0$, for large t and Λ^*_{red} has no base point. It follows that

$$I(t^{n-i}Z^{*(n-i)}, X^{*(i-1)}) \leqslant I(t^{n-i-1}Z^{*(n-i-1)}, tZ^* + F, X^{*(i-1)})$$
$$\leqslant I(t^{n-i-1}Z^{*(n-i-1)}, ((n-1)!+1)t\tau X^*, X^{*(i-1)}).$$

Hence $I(Z^{*(n-i)}, X^{*(i-1)}) \leqslant ((n-1)!+1)\tau I(Z^{*(n-i-1)}, X^{*(i)})$.

When we repeat this, use (a) and note that $Z^{*(n-1)} = \Lambda^{*[n-1]}$,

(b) follows at once.

The problem (I) has been solved in Lemma 4.1. We can now solve the problem (II).

Lemma 4.4

With the same notations and assumptions of Lemmas 3.4 and 4.1, assume further that $\tau(u) > ((n!+1)(n-3))^{\frac{1}{n-1}}$. Let

$$\alpha = d.\tau(u)/[\tau(u)^{n-1} - (n!+1)(n-3)],$$

$$\beta = (du^2 + 2\xi u + \eta)/[\tau(u)^{n-1} - (n!+1)(n-3)].$$

Then $I(Y^*_s - \nu Z^*_s, Z^*_s{}^{(i)}, X^{*(n-i-2)}) < 0$ for $1 \leq i \leq n-2$ if

$$\nu > (n!+1)\{((n-1)!+1)\tau\}^{n-2}(\alpha r + \beta).$$

Proof As usual, let us omit indices s. Let

$$I = I(Y^* - \nu Z^*, Z^{*(i)}, X^{*(n-i-2)}).$$

$$I = I(Y^*, Z^{*(i)}, X^{*(n-i-2)}) - \nu I(Z^{*(i+1)}, X^{*(n-i-2)})$$

$$\leq \{((n-1)!+1)\tau\}^{i} I(K_V + uX, K_V + (u+r)X, X^{(n-2)})$$

$$-(\nu/\{((n-1)!+1)\tau\}^{n-i-2}).(\{\tau(u)^{n-1}/(n!+1)\} - n+3).$$

Therefore, all we have to do is to take ν so that the right hand side of the above strictly negative.

We now define the number ν as follows:

$$\nu = [(n!+1)((n-1)!+1)^{n-1}\tau^{n-1}(\alpha r + \beta)] + 1,$$

where the symbol $[t]$ here is used to denote the largest integer which does not exceed t. Then we get the following result immediately from the above Lemma, Lemma 3.4, Lemma 4.1 and from the formula (**) in § 3.

Proposition 4.1

With the notations of § 3, assume that

$$\chi_s = \chi(D_s; T_{s-1}.D_s) > 0.$$

Then

$$\chi_s < \nu^{n-2}.I(D_s, K_V + (u+r)X, X^{(n-2)}) + 1.$$

By our choice of u and by Lemma 1.2, $I(D_s, K_V + (u+r)X, X^{(n-2)})$ is always non-negative/ When that is so, we get the following corollary.

Corollary

$$\Sigma\chi(D_s;T_{s-1}.D_s) \leqslant \nu^{n-2} I(K_V + uX,\ K_V + (u+r)X,\ X^{(n-2)}) + I(K_V + uX,\ X^{(n-1)}).$$

An analogus formula to (1), § 1, in terms of our linear subsystem Λ of $\Lambda(K_V + uX)$ is valid. $\chi(V,\underline{O}(K_V + (u+r)X))$ is a polynomial of degree n in $u + r$, whose coefficients in absolute values coincide with those of $\chi(V,\underline{O}(mX))$. Hence it is a polynomial of degree n in r with the leading coefficient $d/n!$ and the remaining coefficients are polynomials in u and the coefficients of $\chi(V,\underline{O}(mX))$ and η with integer coefficients. Thus we get the following theorem.

Theorem

Let u be a positive integer, satisfying

$$u \geqslant n + 2,\quad \iota(u) = u + \xi/d > \{(n!+L)(n-3)\}^{1/(n-1)},$$

$$\chi(V,\underline{O}(K_V + uX)) > R^*_n(u),$$

with $d = X^{(n)}$, $\xi = I(K_V,\ X^{(n-1)})$, $R^*_n(x) = R_n(X) + R'_n(x) + R''_n(x)$,

$R_n(x) = d.\ \Sigma_1^{n-1}\ \iota(x)^i$, $R'_n(x) = d.\iota(x)^n/(n! + 1) + d\,\iota(x)$, $R''_n(x) = (d + 1/(n!+1)).\ \iota(x)^{n-1} + n$. Further, let $\eta = I(K_V{}^{(2)}, X^{(n-2)})$, r an arbitrary positive integer and $d_o = d/n!$, $d_1, \dots, d_n$ the coefficients of $\chi(V,\underline{O}(mX))$ as a polynomial in m. Then there is a polynomial $Q(x_o,\dots,x_n,x_{n+1})$ with rational coefficients, which depends only on n and which has degree $n - 1$ in x_{n+1}, such that

$$h^o(V,\underline{O}(rX)) \geq \chi(V,\underline{O}(rX)) - Q(d_o,\dots,d_n,r).$$

§ 5 Appendix

Let V^3 be a non-singular projective variety and X an ample V-divisor. Using the same notations of the foregoing paragraphs, assume that

$$\chi(rX) > R^*_3(r).$$

Then the complete linear system $\Lambda(rX)$ contains a linear subsystem Λ, satisfying the conditions (a),(b),(c) of Lemma 2.6, when $\iota(u)$ is replaced by r and n by 3. In this paragraph, we remove the restriction on the characteristic, and assume that the characteristic of the universal domain is characteristic p. We proved,in "Polarized 3-folds in characteristic p", the following facts. Let

$$\begin{aligned} \rho(X) &= \max(\text{roots of } dx^2 + 2\xi x + \eta = 0) \text{ if there is a non-negative root,} \\ &= 0 \text{ otherwise.} \end{aligned}$$

Let u be a positive integer, satisfying $u \geqslant \max(\rho(X)+1+|\xi|/d, 3)$, then for an arbitrary subvariety D of V of codimension 1,

$$I(K_V + uX + D, D, X) \geqslant 0, \; I(K_V + uX, D, X) \geqslant 0.$$

Using our linear system Λ in the similar way as in § 3, § 4, letting again $W + \sum_1^w D_i$ to be a general divisor of Λ, and

$$T_{s-1} \sim K_V - \Sigma_0^{s-1} D_i ,$$

we consider an exact sequence

$$0 \longrightarrow \underline{O}(T_{s-1} - D_s) \longrightarrow \underline{O}(T_{s-1}) \longrightarrow \underline{O}(D_s;\ T_{s-1}.D_s) \longrightarrow 0.$$

The corresponding cohomology exact sequence yields

$$\chi(T_s) - \chi(T_{s-1}) + \chi(D_s;T_{s-1}.D_s) - h^1(\underline{O}(T_s)) + h^1(\underline{O}(T_{s-1})) \geqslant 0.$$

When we sum these up, we find that

$$(5.1) \quad \chi(K_V - rX) - \chi(K_V) + \Sigma\chi(D_s;T_{s-1}.D_s) - h^1(\underline{O}(K_V - rX)) + h^1(\underline{O}(K_V)) \geqslant 0.$$

As in §§ 3- §§ 5, we estimate $\sum\chi(D_s;T_{s-1}.D_s)$. For this purpose we assume as before that the indices has been chosen so that

$$I(\sum_o^{s-1}D_i,\ D_s,\ X) > 0.$$

We pass to the normalization of D_s and consider the complete linear system determined by the transform of $T_{s-1}.D_s$ (taking T_{s-1} so that no component of the intersection is singular on D_s) and the linear system determined by $Tr_{D_s}\Lambda_{red}$. But in order to simplify notations, we shall identify D_s with its normalization, since it would not confuse our discussions.

Let Y_s be a general divisor of $\Lambda(D_s;\ T_{s-1}.D_s)_{red}$ and Z_s a general divisor of $(Tr_{D_s}\Lambda_{red})_{red}$. By our choice of Λ, the latter linear system is not composed of a pencil. It follows that Z_s is of the form $p^tZ'_s$ with an absolutely irreducible Z'_s and a non-negative integer t. When that is so, the complete linear system determined by Z_s has the property that a general divisor is absolutely irreducible. Therefore, we may assume from the beginning that Z_s itself is absolutely irreducible in the following discussions.

We look for the smallest possible positive integer ν such that

$$I(Y_s - \nu Z_s,\ X) < 0,$$

assuming that $\Lambda(D_s;T_{s-1}.D_s)$ is not empty. We have

$$\begin{aligned}
I(Y_s - \nu Z_s,X) &= I(Y_s,X) - \nu I(Z_s,X)\\
&\leqslant I(K_V - \sum_o^{s-1}D_i,D_s,X) - \nu I(Z_s,X)\\
&\leqslant I(K_V,\ D_s,\ X) - \nu I(Z_s,X)\\
&\lneqq I(K_V + uX,\ D_s,\ X) - \nu I(Z_s,X)\\
&\leqslant I(K_V + uX,\ rX,\ X) - \nu I(Z_s,X)
\end{aligned}$$

by our choice of u.

Assume that $I(Z_s,\ X) < r$. Then

$$3trX.D_s \sim tZ_s + F\ ,\ F > 0$$

by Lemma 3.1 and Lemma 2.6(a version of). It follows that

$$3I(Z_s, trX) \geqslant I(Z_s, tZ_s + F) \geqslant t(Tr_{D_s} \Lambda_{red})^{[2]} \geqslant tr^2/7$$

by Lemma 2.1 and by our version of Lemma 2.6. Thus $I(Z_s, X) \geqslant r/21$ and it is enough to take $\nu \geqslant 21(\xi + du)$.

Assume this time that $I(Z_s, X) \geqslant r$. In this case, it is obvious that $\nu \geqslant (\xi + du)$ is enough. Therefore we conclude that

(5.2) when $\nu = 21(\xi + du)$, $I(Y_s - \nu Z_s, X) < 0$.

Next we look for an upper bound for $I(Z_s, X)$.

$$rI(X^{(2)}, D_s) = I(X, W, D_s) + \Sigma_{i \geqslant 1} I(X, D_i, D_s)$$
$$\geqslant I(Z_s, X) + a_s . I(D_s^{(2)}),$$

where a_s is the coefficient of D_s in the reduced expression for $W + \Sigma_1^w D_i$. As $I(K_V + uX + D_s, D_s, X) \geqslant 0$, $I(K_V + uX, D_s, X) > 0$, it follows that

$$rI(X^{(2)}, D_s) \geqslant I(Z_s, X) - r.(\xi + du)$$
$$\geqslant I(Z_s, X) - r(\xi + du).I(X^{(2)}, D_s).$$

Thus we get

(5.3) $\qquad I(Z_s, X) \leqslant r.(\xi + du + 1).I(X^{(2)}, D_s).$

We can find now an upper bound for $\chi(D_s; T_{s-1}.D_s)$. From (5.3) and Lemma 3.1,

$$3tr(\xi + du + 1).X.D_s \sim tZ_s + F, \; F > 0$$

for large t. Assume that $\Lambda(D_s; T_{s-1}.D_s)$ is not empty. Using the notation of (5.2),

$$\chi(Y_s) \lesssim \nu(\deg Y_s.Z_s + 1) \leqslant 3r\nu(\xi + du + 1)I(Y_s, X) + \nu$$
$$\leqslant 3r\nu(\xi + du + 1)I(T_{s-1}, D_s, X) + \nu$$
$$= 3r\nu(\xi + du + 1)I(K_V - \Sigma_o^{s-1}D_i, D_s, X) + \nu$$
$$\leqslant 3r\nu(\xi + du + 1)I(K_V, D_s, X) + \nu.$$

When K_V above is replaced by $K_V + uX$, the right hand side is strictly positive for all s, and we get

$\chi(Y_s) \leq 61(\xi + du + 1)(\xi + du)r.I(K_V + uX, D_s, X) + 21(\xi + du).$
From this, we get

(5.5) $$\sum \chi(Y_s) \leq 61(\xi + du + 1)(\xi + du)^2 r^2 + 21(\xi + du).dr,$$

as the number of components in $W + \sum_1^w D_i$ is at most dr.

From this and from (5.1), we get

$$h^2(V, \underline{O}(rX)) \leq \chi(K_V - rX) - p_g + h^1(V, \underline{O}(K_V)) + 61\lambda^2(\lambda + 1)r^2 + 21\lambda dr,$$
$$\lambda = \xi + du.$$

Using this, we can conclude that when $\chi(rX)$ is large enough, and when (U, Y) is an an arbitrary deformation of (V, X) in characteristic p, $\chi(rY)$ is large enough. But in characteristic $p > 0$, definitions and details have to be worked out a little bit more carefully than the case of characteristic 0. For details, see the author's paper (loc. cit.).